A GCD–LCM Game:
From IMO 2026 to Counting Moves

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Abstract

Starting from Problem 1 of IMO 2026, we study the number of moves in the process that replaces two integers $m, n > 1$ by $\gcd(m,n)$ and $\operatorname{lcm}(m,n)/\gcd(m,n)$. For two initial entries, the number of moves is determined by subtractive Euclidean algorithms on the exponents of the prime factors. For coprime integers $A, B > 1$, we determine all possible numbers of moves when the initial entries belong to $\{A, B, AB\}$. These numbers are consecutive integers, and we give explicit formulas for the smallest and largest values. The board (A^2B, A^2B, AB^2), however, has possible lengths $\{3, 5, 6, 7\}$. For each integer $\ell \geq 4$, we also construct a board admitting plays of lengths $\ell - 1$ and $\ell + 1$, but no play of length ℓ.

Keywords. Greatest common divisor; least common multiple; Euclidean algorithm; invariant; number of moves.

1 Introduction

Problem 1 of IMO 2026 concerns a board containing 2026 integers greater than 1. At each move, we choose two entries $m, n > 1$, erase them, and write in their places

$$\gcd(m,n) \quad \text{and} \quad \frac{\operatorname{lcm}(m,n)}{\gcd(m,n)}. \tag{1}$$

Here $\gcd(m,n)$ and $\operatorname{lcm}(m,n)$ denote the greatest common divisor (gcd) and least common multiple (lcm) of m and n, respectively. The chosen entries must occupy different positions, but they may have the same value. Entries equal to 1 cannot be chosen. We continue until fewer than two entries greater than 1 remain. Such a complete sequence of moves will be called a *play*, and its *length* is its number of moves.

The problem asks us to prove that every play ends with exactly one entry greater than 1, and that this entry is independent of the pairs chosen; see [1]. The conclusion remains valid when 2026 is replaced by any integer $r \geq 2$. The invariant involving the exponents of the prime factors, and the formula for the terminal value, appear in the solutions of Chen [2] and Bado [3]. The decreasing quantity used to prove termination in Section 2 also appears in [3].

Every play from $(2, 2, 6)$ ends with the same number. Can you find two plays of different lengths? Try starting once with the two 2s and once with 2 and 6.

We omit the 1s and ignore order, but keep repetitions. Each arrow is one move:

$$(2,2,6) \longmapsto (2,6) \longmapsto (2,3) \longmapsto 6,$$
$$(2,2,6) \longmapsto (2,2,3) \longmapsto (2,6) \longmapsto (2,3) \longmapsto 6.$$

Both plays end at 6, the first after three moves and the second after four.

Which lengths are possible for a given initial board? Let $\mathcal{L}(\mathbf{a})$ denote the set of all possible lengths of plays from $\mathbf{a} = (a_1, \dots, a_r)$. The two plays above show that $3, 4 \in \mathcal{L}(2, 2, 6)$; they do not rule out other lengths. We determine the entire set $\mathcal{L}(\mathbf{a})$ in several cases. For a board consisting of a single entry $a > 1$, we put $\mathcal{L}(a) = \{0\}$, since no move is needed.

The case of two entries leads to the subtractive Euclidean algorithm. Its number of steps is the sum of the quotients in the ordinary Euclidean algorithm; this is a classical result, for which see [4]. The length of the subtractive algorithm is also studied in [5]. Theorem 3.3 combines the step counts for the individual prime factors to determine the number of moves in the game.

For boards with more entries, we consider the case in which every entry belongs to $\{A, B, AB\}$, where $A, B > 1$ and $\gcd(A, B) = 1$. Every new entry greater than 1 still belongs to this set. Theorem 4.1 determines all possible lengths for these boards. Theorem 5.1 then exhibits a board whose possible lengths are not consecutive integers, and Lemma 5.3 extends this example to an infinite family.

Notation for exponents. For a prime p and a positive integer t, let $v_p(t)$ denote the exponent of p in the prime factorization of t. If p does not divide t, then $v_p(t) = 0$. For example, $72 = 2^3 \cdot 3^2$ gives $v_2(72) = 3$, $v_3(72) = 2$, and $v_5(72) = 0$. Put $\Omega(t) = \sum_p v_p(t)$, where the sum is over all primes. Only finitely many terms are nonzero. Thus $\Omega(t)$ counts the prime factors of t, with repetitions counted separately; for example, $\Omega(72) = 5$.

We use $v_p(1) = \Omega(1) = 0$. Since $v_p(ab) = v_p(a) + v_p(b)$, we have $\Omega(ab) = \Omega(a) + \Omega(b)$. Also, $\gcd(u, 0) = u$ for $u \geq 0$, so adjoining zero entries does not change the gcd of a list of numbers. We use the convention $\gcd(0, \dots, 0) = 0$.

2 The terminal value and bounds on the number of moves

We recall the argument using the exponents of prime factors so that the reader need not consult another solution. The main arguments in the following theorem already appear in [2, 3].

Theorem 2.1. *Let $a_1, \dots, a_r > 1$ be integers, where $r \geq 2$. Starting from $(a_1, \dots, a_r)$, the game always terminates after finitely many moves with exactly one entry $M > 1$. Set $P = a_1 \cdots a_r$. Then*

$$M = \prod_{p \mid P} p^{\gcd(v_p(a_1), \dots, v_p(a_r))}, \tag{2}$$

where the product is over the prime divisors of P. In particular, M is independent of the pairs chosen.

Proof. We first prove termination. In this proof, we retain the entries equal to 1 when writing the board as $\mathbf{b} = (b_1, \dots, b_r)$. Set $P(\mathbf{b}) = b_1 \cdots b_r$, and let $C(\mathbf{b})$ be the number of entries greater than 1.

Suppose a move selects $m, n > 1$, and set $d = \gcd(m, n)$. Since $\gcd(m, n)\,\mathrm{lcm}(m, n) = mn$, the new entries are d and mn/d^2. The removed entries have product mn, whereas the new entries have product mn/d. All other entries remain unchanged, so the product of the entries on the board is divided by d. By the additivity of Ω, the quantity $\Omega(P(\mathbf{b}))$ decreases by exactly $\Omega(d)$.

If $d = 1$, the new pair is $(1, mn)$, so C decreases by one. If $d > 1$, then $\Omega(P(\mathbf{b}))$ decreases by at least one, while C does not increase.

Thus the nonnegative integer

$$\Phi(\mathbf{b}) = \Omega(P(\mathbf{b})) + C(\mathbf{b})$$

decreases by at least one at each move. A nonnegative integer cannot decrease in this way indefinitely, so the game must terminate.

At termination, at most one entry is greater than 1; if there were two such entries, they could still be selected. On the other hand, the product of the two new entries is always $\operatorname{lcm}(m,n) > 1$, so every move leaves at least one entry greater than 1. Hence the terminal board has the form $(M, 1, \dots, 1)$, where $M > 1$.

To determine M, we use an *invariant*, that is, a quantity unchanged by every move. Fix a prime p, and set $u = v_p(m)$ and $v = v_p(n)$. The exponent of p in the gcd is $\min(u,v)$, and its exponent in the lcm is $\max(u,v)$. Thus the exponents of p in the two new entries are

$$\min(u,v) \quad \text{and} \quad \max(u,v) - \min(u,v) = |u-v|,$$

respectively. We have $\gcd(u,v) = \gcd(\min(u,v), |u-v|)$: if $u \geq v$, this is the identity $\gcd(u,v) = \gcd(v, u-v)$. The exponents at positions not selected remain unchanged. It follows that

$$\gcd(v_p(b_1), \dots, v_p(b_r))$$

is invariant. On the terminal board, the exponents are $v_p(M), 0, \dots, 0$, whose gcd is $v_p(M)$. Hence

$$v_p(M) = \gcd(v_p(a_1), \dots, v_p(a_r)).$$

Each new entry divides mn, so a prime absent from the initial board cannot occur later. Equation (2) follows. □

For example, on the board $(12, 12, 18)$, the exponents of 2 are $(2,2,1)$, and those of 3 are $(1,1,2)$. Both triples have gcd 1. Since no other prime occurs, the terminal value is always $2 \cdot 3 = 6$.

The quantities C and Φ also yield lower and upper bounds on the number of moves. A lower bound is a number below or equal to every possible length; an upper bound is defined in the same way, with the inequality reversed.

Proposition 2.2. *Retain the notation of Theorem 2.1. If a play has N moves, then*

$$r - 1 \leq N \leq \Omega(P) - \Omega(M) + r - 1. \tag{3}$$

Equality holds on the left if and only if every selected pair consists of equal or coprime entries. Equality holds on the right if and only if every selected pair consists of coprime entries, or of distinct entries whose gcd is prime.

Proof. For a move selecting m, n, set $d = \gcd(m,n)$. If $d = 1$, the new pair is $(1, mn)$. If $m = n$, the new pair is $(m, 1)$. In both cases, C decreases by one. In the remaining case, $d > 1$ and $m \neq n$. Then $\operatorname{lcm}(m,n) > d$, so both new entries are greater than 1, and C is unchanged.

Thus each move decreases C by at most one. Since C decreases from r to 1, at least $r-1$ moves are required. The number of moves is $r-1$ if and only if each move decreases C by one, which occurs precisely when every selected pair consists of equal or coprime entries.

For the upper bound, let Δ denote the decrease in one move, namely, the value before the move minus the value after it. The calculation in the proof of Theorem 2.1 yields the following table.

Case	$\Delta\Omega(P(\mathbf{b}))$	ΔC	$\Delta\Phi$
$d = 1$	0	1	1
$m = n$	$\Omega(m)$	1	$\Omega(m) + 1$
$d > 1, m \neq n$	$\Omega(d)$	0	$\Omega(d)$

Each decrease in Φ is at least 1. Its initial and final values are $\Omega(P) + r$ and $\Omega(M) + 1$, respectively, so

$$N \leq (\Omega(P) + r) - (\Omega(M) + 1).$$

Equality holds if and only if every move decreases Φ by exactly one. In the second row of the table, $\Omega(m) + 1 \geq 2$ since $m > 1$. In the third row, $\Omega(d) = 1$ if and only if d is prime. These observations give exactly the stated condition. □

3 The number of moves from two entries

If the initial board contains only two entries $m, n > 1$, then there is exactly one legal pair at every move before termination. Thus $\mathcal{L}(m, n)$ has a single element, which we denote by $N(m, n)$. The transformation of the exponents in the preceding proof leads to the subtractive Euclidean algorithm.

At each step of this algorithm, we retain the smaller number and replace the larger number by their difference. We stop when one number is 0. More precisely, set $F(u, v) = (\min(u, v), |u - v|)$ for nonnegative integers u, v. We write $F^j(u, v)$ for the pair obtained after j successive applications of F, with $F^0(u, v) = (u, v)$.

Lemma 3.1. *Let u, v be nonnegative integers with $(u, v) \neq (0, 0)$. Let $s(u, v)$ be the number of applications of F until one component first becomes 0; if $uv = 0$ initially, set $s(u, v) = 0$. Then $s(u, v)$ is finite.*

Set $g = \gcd(u, v)$ and $s = s(u, v)$. If $s > 0$, then

$$F^s(u, v) = (g, 0) \quad \text{and} \quad F^t(u, v) = (0, g) \text{ for all } t \geq s + 1.$$

If $s = 0$, then $F^t(u, v) = (0, g)$ for all $t \geq 1$.

Proof. While both components are positive, their sum decreases at each step, since $\min(u, v) + |u - v| = \max(u, v) < u + v$. This sum is a nonnegative integer, so one component must become 0 after finitely many steps.

The gcd of the two components is preserved by F. If $s > 0$, both components are positive immediately before step s. The first component after this step is the smaller of those two positive numbers, so it remains positive. Hence the second component is the one that becomes 0. The resulting pair has gcd g and must therefore be $(g, 0)$.

We have $F(g, 0) = (0, g)$ and $F(0, g) = (0, g)$. Thus one further application of F gives $(0, g)$, which remains fixed thereafter. If $s = 0$, the initial pair is $(g, 0)$ or $(0, g)$, so the same two identities prove the assertion. □

Under this convention, the transformation $(g, g) \mapsto (g, 0)$ counts as one step. This explains why the formula below includes the quotient in the final division. The formula is a classical result on the subtractive Euclidean algorithm; see [4].

Lemma 3.2. *Let u, v be positive integers. If $q_1, \dots, q_t$ are the quotients in the Euclidean algorithm applied to* $\max(u, v)$ *and* $\min(u, v)$*, including the quotient in the final division with remainder* 0*, then*

$$s(u, v) = q_1 + \cdots + q_t.$$

Proof. Since $F(u, v) = F(v, u)$, we have $s(u, v) = s(v, u)$ and may assume that $u \geq v$. Write $u = qv + w$, where $0 \leq w < v$. Each of the first q steps subtracts v from the other number. Indeed, after $j < q$ subtractions, that number is $u - jv \geq v$; after exactly q subtractions, it is w. Hence the pair after q steps is (v, w).

If $w = 0$, the algorithm terminates after exactly q steps. If $w > 0$, a further $s(v, w)$ steps are required. In both cases,

$$s(u, v) = q + s(v, w), \qquad \text{where } s(v, 0) = 0.$$

Applying this identity successively to the divisions in the Euclidean algorithm, each quotient counts the corresponding subtractions. The total number of steps is therefore $q_1 + \cdots + q_t$. □

For example, the divisions $8 = 2 \cdot 3 + 2$, $3 = 1 \cdot 2 + 1$ and $2 = 2 \cdot 1$ yield $s(8, 3) = 2 + 1 + 2 = 5$. The five subtraction steps are

$$(8, 3) \longmapsto (3, 5) \longmapsto (3, 2) \longmapsto (2, 1) \longmapsto (1, 1) \longmapsto (1, 0).$$

Returning to the game, an entry t equals 1 if and only if $v_p(t) = 0$ for every prime p. It is not enough that each pair of exponents has a zero component: those zero components must all belong to the same entry on the board. For example, in the pair $(2, 3)$, the exponent of 2 is 0 in the second entry, while the exponent of 3 is 0 in the first; both entries are still greater than 1.

Theorem 3.3. *Let $m, n > 1$ be integers. For each prime $p \mid mn$, set*

$$s_p = s(v_p(m), v_p(n)) \quad \textit{and} \quad K = \max_{p \mid mn} s_p.$$

Then

$$N(m, n) = \begin{cases} K, & \textit{if } K > 0 \textit{ and } s_p = K \textit{ for every prime } p \mid mn, \\ K + 1, & \textit{otherwise}. \end{cases} \tag{4}$$

Proof. At each move, we place the gcd in the first position. Set $(m_0, n_0) = (m, n)$, and define the subsequent pairs by

$$m_{j+1} = \gcd(m_j, n_j) \quad \text{and} \quad n_{j+1} = \frac{m_j n_j}{\gcd(m_j, n_j)^2} \qquad \text{for } j \geq 0.$$

These formulas still make sense when one entry equals 1. In counting the moves of the game, we stop at the first index j for which $m_j = 1$ or $n_j = 1$; the later pairs are used only to follow the formulas.

For each prime $p \mid mn$, set $g_p = \gcd(v_p(m), v_p(n))$. At least one of the two exponents is positive, so $g_p > 0$. The transformation of the exponents yields

$$(v_p(m_{j+1}), v_p(n_{j+1})) = F(v_p(m_j), v_p(n_j)).$$

Thus Lemma 3.1 describes the positions of the zero exponents at every move. No prime other than a prime divisor of mn can occur.

If $K = 0$, then $s_p = 0$ for every prime $p \mid mn$, which means that each prime divides only one of m, n. Hence $\gcd(m, n) = 1$, and the first move gives $(1, mn)$. It follows that $N(m, n) = 1 = K + 1$.

Suppose that $K > 0$. There is a prime p for which $s_p = K$. Before move K, both exponents of p are positive, so both entries on the board are greater than 1. Thus the game cannot terminate before move K.

If $s_p = K$ for every prime $p \mid mn$, Lemma 3.1 gives the exponent pair $(g_p, 0)$ after move K for every such p. Thus $v_p(n_K) = 0$ for every prime p, so $n_K = 1$ and $N(m, n) = K$.

It remains to consider the case where $s_q < K$ for some prime q. If $s_q > 0$, then $K \geq s_q + 1$; if $s_q = 0$, then $K \geq 1$. In either case, Lemma 3.1 gives the exponent pair $(0, g_q)$ for q after move K. Hence $q \mid n_K$, so $n_K > 1$. At the same time, a prime p for which $s_p = K$ still divides m_K, so $m_K > 1$.

After move K, the pair of exponents for each prime factor has at least one zero component. Thus $\gcd(m_K, n_K) = 1$. The next move gives $(1, m_K n_K)$ and terminates the game. Hence $N(m, n) = K + 1$. □

Remark. For a prime p and positive integers a, b, there is only one pair of exponents to consider. Hence $N(p^a, p^b) = s(a, b)$, and the number of moves is the sum of the quotients in the Euclidean algorithm applied to a, b.

Example 3.4. For $(m, n) = (6, 12)$, the exponent pairs for 2 and 3 are $(1, 2)$ and $(1, 1)$, respectively, so $s_2 = 2$ and $s_3 = 1$. Since $K = 2$ but $s_2 \neq s_3$, we have $N(6, 12) = 3$. The following table lists the exponent pairs in the same order as the two entries on the board.

Moves made	Board	Exponents of 2	Exponents of 3
0	$(6, 12)$	$(1, 2)$	$(1, 1)$
1	$(6, 2)$	$(1, 1)$	$(1, 0)$
2	$(2, 3)$	$(1, 0)$	$(0, 1)$
3	$(1, 6)$	$(0, 1)$	$(0, 1)$

After two moves, each exponent pair contains a 0, but the zeros occur in different positions. Only after three moves are all exponents in the first position equal to 0.

For $(m, n) = (12, 18)$, the exponent pairs for 2 and 3 are $(2, 1)$ and $(1, 2)$, and both require two subtraction steps. Here $N(12, 18) = 2$, and the play is

$$(12, 18) \longmapsto (6, 6) \longmapsto (6, 1).$$

4 Classification for boards with three types of entries

Let $A, B > 1$ be coprime integers. Consider a board with x entries equal to A, y entries equal to B, and z entries equal to AB. We denote its set of possible lengths by $\mathcal{L}_{A,B}(x, y, z)$. A count of 0 means that no entry of that type is present. For example, when $A = 2$ and $B = 3$, the board $(2, 2, 6)$ corresponds to $(x, y, z) = (2, 0, 1)$.

Every entry greater than 1 produced by a move still belongs to $\{A, B, AB\}$. For instance, since $\gcd(A, AB) = A$ and $\operatorname{lcm}(A, AB) = AB$, the pair (A, AB) is replaced by (A, B). The number of entries equal to A is unchanged, the number equal to B increases by one, and the number equal to AB decreases by one. The table in the proof below lists all possible moves.

Theorem 4.1. *Let $A, B > 1$ be coprime integers, and let $x, y, z \geq 0$ be integers with $r = x + y + z \geq 2$. If exactly one of x, y, z is nonzero, then*

$$\mathcal{L}_{A,B}(x, y, z) = \{r - 1\}.$$

If at least two of x, y, z are nonzero, put

$$\varepsilon = \begin{cases} 0, & \text{if } x > 0 \text{ and } y > 0, \\ 1, & \text{if } z > 0 \text{ and } xy = 0. \end{cases}$$

Then the terminal value is AB, and

$$\mathcal{L}_{A,B}(x, y, z) = \{r - 1 + \varepsilon, \dots, 2r + z - 3\}. \tag{5}$$

Thus every integer between these two endpoints is the length of a play, and no other length is possible.

Proof. If only one type of entry is present, each move chooses two equal entries and replaces them by one entry of the same value and one entry equal to 1. The number of entries greater than 1 decreases by exactly one at each move, so the length is $r - 1$.

Suppose that at least two types are present. For each prime $p \mid A$, the exponents of p on the initial board are either 0 or $v_p(A)$, and at least one equals $v_p(A)$. Their greatest common divisor is therefore $v_p(A)$. The same argument applies to the primes dividing B. Since $\gcd(A, B) = 1$, formula (2) shows that the terminal value is AB.

Counting moves and obtaining bounds. To distinguish the current counts from the initial counts x, y, z, write (i, j, k) for the numbers of entries of the three types on an intermediate board. Put $C = i + j + k$ and $W = i + j + 2k$. Thus C counts the entries greater than 1, while W assigns weight 1 to each entry equal to A or B and weight 2 to each entry equal to AB. Initially, $C = r$ and $W = W_0$, where $W_0 = x + y + 2z$. At the end only AB remains, so $C = 1$ and $W = 2$.

In the following table, ΔC and ΔW denote the value before the move minus the value after it. A pair of equal entries can be chosen only if that value occurs at least twice; every other pair requires at least one entry of each type involved.

Chosen pair	New pair	New counts	ΔC	ΔW
(A, A)	$(A, 1)$	$(i - 1, j, k)$	1	1
(B, B)	$(B, 1)$	$(i, j - 1, k)$	1	1
(AB, AB)	$(AB, 1)$	$(i, j, k - 1)$	1	2
(A, B)	$(1, AB)$	$(i - 1, j - 1, k + 1)$	1	0
(A, AB)	(A, B)	$(i, j + 1, k - 1)$	0	1
(B, AB)	(B, A)	$(i + 1, j, k - 1)$	0	1

Let N be the total number of moves, and let H count the moves that choose (A, AB) or (B, AB). According to the table, H moves leave C unchanged, and the remaining $N - H$ moves decrease C by one. Since C decreases from r to 1, we have $N - H = r - 1$, or

$$N = r - 1 + H. \tag{6}$$

The total decrease in W is $W_0 - 2$. Each move counted in H contributes exactly one to this decrease, and every other move contributes a nonnegative amount. Hence $H \leq W_0 - 2$.

If $x, y > 0$, we can attain $H = 0$. First choose pairs (A, A) until only one entry equal to A remains, and do the same for the entries equal to B. Next choose (A, B), producing

one entry equal to AB and one equal to 1. Every entry greater than 1 now equals AB; keep choosing pairs (AB, AB) until only one such entry remains. This play never chooses (A, AB) or (B, AB), so $H = 0$.

If $y = 0$ and $x, z > 0$, a sequence with no move counted in H can only choose (A, A) or (AB, AB). Such moves always leave at least one entry equal to A and at least one equal to AB, so the game cannot end. Therefore $H \geq 1$. The case $x = 0$ and $y, z > 0$ is similar. We have proved $\varepsilon \leq H \leq W_0 - 2$.

Attaining every integer between the bounds. We use induction on the integer W_0, considering all boards with at least two types of entries. If $W_0 = 2$, the board must be (A, B), and $H = 0$. Suppose that $W_0 > 2$ and that the assertion holds for every board in this class whose initial weight is less than W_0.

If $z > 0$, at least one of x, y is positive. Interchanging A, B and simultaneously interchanging x, y if necessary, we may assume that $x > 0$. Choosing (A, AB) changes the counts to $(x, y + 1, z - 1)$. This board contains both A and B and has weight $W_0 - 1$. By the induction hypothesis, the number of subsequent moves choosing (A, AB) or (B, AB), denoted by H', attains every value $0, 1, \dots, W_0 - 3$. The move just made is also counted in H, so $H = 1 + H'$ attains every value $1, \dots, W_0 - 2$. If the initial board contains both A and B, we have already constructed a play with $H = 0$.

If $z = 0$, then $x, y > 0$ and $x + y = W_0 > 2$. Thus at least one of x, y is at least 2; after interchanging the two types if necessary, assume that $x \geq 2$. Choose (A, B) and then (A, AB). The counts change as follows:

$$(x, y, 0) \longmapsto (x - 1, y - 1, 1) \longmapsto (x - 1, y, 0).$$

The second move is allowed because the first leaves $x - 1 \geq 1$ entries equal to A. Of these two moves, only the second is counted in H. The resulting board still contains both A and B and has weight $W_0 - 1$. The induction hypothesis allows H' to attain every value $0, \dots, W_0 - 3$, so $H = 1 + H'$ attains every value $1, \dots, W_0 - 2$. The value $H = 0$ has already been treated.

Consequently, every integer from ε to $W_0 - 2$ is attained. In view of (6), the minimum length is $r - 1 + \varepsilon$, and the maximum length is

$$r - 1 + (W_0 - 2) = r - 1 + (r + z - 2) = 2r + z - 3.$$

Every integer between these two values is attained, which proves (5). □

Corollary 4.2. *For every integer $r \geq 2$ and coprime integers $A, B > 1$, we have*

$$\mathcal{L}(\underbrace{A, \dots, A}_{r-1 \text{ entries}}, AB) = \{r, \dots, 2r - 2\}.$$

Proof. Here $(x, y, z) = (r - 1, 0, 1)$, so $\varepsilon = 1$. Substitution in the two endpoints in Theorem 4.1 yields $r - 1 + 1 = r$ and $2r + 1 - 3 = 2r - 2$. □

For example, the board $(2, 2, 6)$ has $(x, y, z) = (2, 0, 1)$, so its set of possible lengths is exactly $\{3, 4\}$. The two plays in the introduction attain these values; the theorem also rules out every other length. The board $(2, 3, 6)$ has $(x, y, z) = (1, 1, 1)$, so its set of possible lengths is $\{2, 3, 4\}$.

Remark. A positive integer is *squarefree* if it is not divisible by p^2 for any prime p; for example, 6 is squarefree, whereas 12 is divisible by 2^2.

Let p, q be distinct primes. Every squarefree integer greater than 1 with no prime divisors other than p, q belongs to $\{p, q, pq\}$. Thus Theorem 4.1 classifies all boards formed

from such integers. If at least two types of entries are present, then $\Omega(P) = x + y + 2z = r + z$ and $M = pq$. The upper bound in Proposition 2.2 is $(r + z) - 2 + r - 1 = 2r + z - 3$, which is exactly the maximum length attained above.

5 The possible lengths need not be consecutive

For the boards considered above, the possible lengths include every integer between the minimum and maximum. The following example shows that this need not hold for an arbitrary board. To determine all possible lengths, we consider every type of pair that can be chosen on the first move. Once the resulting board is known, the total length is one plus the length of a play from that board.

Theorem 5.1. *Let $A, B > 1$ be coprime integers. Then*

$$\mathcal{L}(A^2B, A^2B, AB^2) = \{3, 5, 6, 7\}. \tag{7}$$

Proof. Set $X = AB$, $U = A^2B$, and $V = AB^2$. Since $\gcd(A, B) = 1$, we have $\gcd(U, V) = X$ and $\operatorname{lcm}(U, V) = X^2$. Thus the pair (U, V) is replaced by (X, X). Also, $X \mid U$ and $U/X = A$, so the pair (X, U) is replaced by (X, A). The plays from these two pairs are

$$(U, V) \longmapsto (X, X) \longmapsto X,$$
$$(X, U) \longmapsto (X, A) \longmapsto (A, B) \longmapsto X.$$

Each play is uniquely determined, since at every stage before termination there are only two entries greater than 1. Hence $N(U, V) = 2$ and $N(X, U) = 3$.

By Theorem 4.1, with $(x, y, z) = (1, 0, 2)$, the possible lengths from (A, X, X) are $\{3, 4, 5\}$.

Consider the board (X, X, U). If the first move selects the two entries X, the resulting board is (X, U), so the total length is $1 + 3 = 4$. If the first move selects X and U, this pair is replaced by (X, A), while the unselected entry X remains unchanged. The resulting board is (A, X, X). The possible remaining lengths are exactly $3, 4, 5$, so the possible total lengths are exactly $4, 5, 6$. These are all possible types of first move. Therefore

$$\mathcal{L}(X, X, U) = \{4\} \cup \{4, 5, 6\} = \{4, 5, 6\}.$$

Finally, consider (U, U, V). Selecting the two entries U on the first move leaves (U, V), so the total length is $1 + 2 = 3$. Selecting U and V on the first move leaves (X, X, U), so the possible total lengths are exactly $5, 6, 7$. There is no other type of pair to select, which proves (7). □

Example 5.2. Take $A = 2$ and $B = 3$. Then $\mathcal{L}(12, 12, 18) = \{3, 5, 6, 7\}$. Plays of lengths three and five are

$$(12, 12, 18) \longmapsto (12, 18) \longmapsto (6, 6) \longmapsto 6,$$
$$(12, 12, 18) \longmapsto (12, 6, 6) \longmapsto (12, 6) \longmapsto (6, 2) \longmapsto (2, 3) \longmapsto 6.$$

There is no play of length four, as follows from the exhaustive analysis of the first move in the proof above. Three is the smallest number of initial entries for which this phenomenon can occur, since a board with two entries has only one possible length.

For boards of the form in the next lemma, we can construct a new board whose possible lengths are obtained by adding the same integer k to every possible length of the original board. For a set of integers S, write $k+S = \{k+n : n \in S\}$; this is the set obtained by adding k to each element of S. For example, $2 + \{3, 5\} = \{5, 7\}$. The lemma does not require a coprimality assumption.

Lemma 5.3. *Let $A, B > 1$, $s \geq 1$, and $k \geq 0$ be integers. Then*

$$\mathcal{L}(\underbrace{A, \ldots, A}_{s \text{ entries}}, A^k B) = k + \mathcal{L}(\underbrace{A, \ldots, A}_{s \text{ entries}}, B). \tag{8}$$

Proof. For $k = 0$, the two sides are the same set of possible lengths. Suppose $k > 0$. Mark the entry $A^k B$ to distinguish it from the other entries. While the marked entry has value $A^j B$, with $j \geq 1$, it is greater than A, and every unmarked entry greater than 1 equals A. Hence every move during this part of the play has one of the two forms

$$(A, A) \longmapsto (A, 1) \quad \text{or} \quad (A, A^j B) \longmapsto (A, A^{j-1} B).$$

In the second case, $A \mid A^j B$, so the gcd is A; we retain the mark on the entry $A^{j-1} B$.

At least one unmarked entry A always remains. Indeed, a move of the first type requires two such entries and leaves one, while a move of the second type preserves their number. Thus, while $j > 0$, there are at least two entries greater than 1, so the play has not ended. By Theorem 2.1, every play ends after finitely many moves. The marked entry must therefore reach the value B. Only moves of the second type decrease j, each by exactly one. Consequently, exactly k such moves occur before the marked entry first becomes B.

Consider any play of length N from the board on the left of (8). Suppose that t moves selecting two unmarked entries A occur before the marked entry becomes B. At the moment it reaches B, the board has $s - t$ unmarked entries A and one marked entry B.

Now start a new play from s entries A and one marked entry B. Repeat those t moves selecting two entries A. They remain legal, since the moves involving the marked entry in the original play did not change the number of unmarked entries A. After these t moves, the new board also has $s - t$ entries A and the marked entry B. Continue with the remaining moves of the original play. The new play has length $N - k$, since exactly the k moves that decreased j have been omitted. This argument also applies when $B = A$, because the marked entry is distinguished from the other entries A.

Conversely, suppose a play of length n starts from s entries A and one entry B. Start instead with the marked entry $A^k B$, and use one of the s entries A to make k consecutive moves of the second type. Each move retains the selected entry A, so all k moves are legal. The marked entry now equals B, and we append the given play of length n. The total length is $k + n$.

These two constructions prove both inclusions in (8). □

Corollary 5.4. *Let $A, B > 1$ be coprime integers, and set $U = A^2 B$ and $V = AB^2$. For every integer $k \geq 0$,*

$$\mathcal{L}(U, U, U^k V) = \{k + 3, k + 5, k + 6, k + 7\}.$$

Consequently, for every integer $\ell \geq 4$, there is a board with three entries that admits plays of lengths $\ell - 1$ and $\ell + 1$, but no play of length ℓ.

Proof. Apply Lemma 5.3 with $s = 2$ and with U, V in place of A, B. Then Theorem 5.1 yields

$$\begin{aligned} \mathcal{L}(U, U, U^k V) &= k + \mathcal{L}(U, U, V) \\ &= k + \{3, 5, 6, 7\} \\ &= \{k + 3, k + 5, k + 6, k + 7\}. \end{aligned}$$

For $\ell \geq 4$, take $k = \ell - 4 \geq 0$. The set above becomes $\{\ell - 1, \ell + 1, \ell + 2, \ell + 3\}$, which contains $\ell - 1$ and $\ell + 1$ but not ℓ. □

For example, take $A = 2$, $B = 3$, and $k = 2$. Then $U = 12$, $V = 18$, and $U^k V = 2592$, so $\mathcal{L}(12, 12, 2592) = \{5, 7, 8, 9\}$.

6 Two further questions

The example $(12, 12, 18)$ uses numbers divisible by the square of a prime. We may ask what happens when all initial entries are squarefree, meaning that no prime square divides any of them. This condition is preserved by every move: the exponent of each prime is either 0 or 1, and replacing (u, v) by $(\min(u, v), |u - v|)$ leaves both exponents in $\{0, 1\}$.

In this case, a number is determined by its set of prime divisors. If two numbers correspond to sets S, T, their gcd corresponds to $S \cap T$, while the second new entry corresponds to

$$S \triangle T = (S \setminus T) \cup (T \setminus S).$$

The set $S \triangle T$ is called the *symmetric difference*: it consists of the elements belonging to exactly one of S, T. For example, the pair $(6, 15)$ is replaced by $(3, 10)$. The corresponding sets of prime divisors change from $\{2, 3\}, \{3, 5\}$ to $\{3\}, \{2, 5\}$.

Try it yourself.

(a) Starting from $(6, 10, 15)$, how many moves can a play take? Find all possible lengths and explain why your list is complete.

(b) Write $6 = 2 \cdot 3$, $10 = 2 \cdot 5$, and $15 = 3 \cdot 5$. Replace $2, 3, 5$ by distinct primes p, q, r, respectively. What does your argument in part (a) tell you about $\mathcal{L}(pq, pr, qr)$?

Hint. Which number occurs twice on the board after the first move?

Each entry of (pq, pr, qr) is the product of two of the three primes. We now ask what happens if we assume only that every initial entry is squarefree.

Question 6.1. If all initial entries are squarefree, must $\mathcal{L}(\mathbf{a})$ consist of consecutive integers?

Theorem 4.1 answers this affirmatively when the product of the initial entries has at most two distinct prime divisors. The question asks what happens when more distinct prime divisors occur.

For arbitrary boards, Corollary 5.4 produces an integer ℓ that is not a possible length, although both $\ell - 1$ and $\ell + 1$ are possible lengths. The integer ℓ can be arbitrarily large, but this family omits only one integer between two attainable lengths. This leads to a question about omitting several consecutive integers.

Question 6.2. For every integer $h \geq 1$, is there a board $\mathbf{a}$ such that $\mathcal{L}(\mathbf{a})$ contains two elements $u < v$ with $v - u - 1 \geq h$, but contains no integer strictly between u and v?

Here $v - u - 1$ is the number of integers $u + 1, \ldots, v - 1$. The question requires both u and v to be attainable lengths, with at least h consecutive unattainable lengths between them. Lemma 5.3 adds the same integer to every possible length and therefore preserves the differences $v - u$. Thus it does not answer the question of whether the number of omitted integers can increase without bound.

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