A Random Tunnel Network

General Round, Problem 10 · HMMT November 2026

English solution · November 7, 2026

Original HMMT problem set. Problem 10 was proposed by Bryan Guo.

A directed graph solution

Let $n \geq 2$ and model the cities as the vertices of a complete graph. Each edge represents a proposed tunnel, and the costs $1, \ldots, \binom{n}{2}$ are assigned to the edges by a uniformly random permutation. Write $c(uv)$ for the cost of edge uv. Draw an arrow $u \to v$ when uv is the cheapest edge incident to u. Each vertex has exactly one outgoing arrow. The built network consists of the underlying undirected edges of these arrows.

Follow the arrows from any starting city. If we have just traversed $u \to v$ and the next arrow is $v \to w$ with $w \neq u$, then $c(vw) < c(uv)$ because uv was also available to v. As there are only finitely many costs, these strict decreases eventually end. At that step v points back to u, and subsequent steps alternate between the two cities.



A six-city example. Give the edges shown the labeled costs and assign 5 through 15 to the remaining edges in any order. Starting at D, the costs along the arrows are $4, 3, 1$ before the walk begins to alternate between A and B. The two components contain the pairs A, B and E, F, respectively.

The endpoints of every built tunnel reach the same pair of cities that choose each other, because at least one endpoint points to the other. Thus all cities in a connected component reach the same pair. Conversely, cities that reach the same pair are connected by their paths to it. Hence **each connected component contains exactly one pair of cities that choose each other**.

Counting with indicators

Let C_n be the number of connected components. For each city u, let $J_u = 1$ if u belongs to a pair of cities that choose each other and $J_u = 0$ otherwise. Each component contributes exactly two cities to this count, so

$$2C_n = \sum_u J_u.$$

Fix distinct cities u, v. They choose each other precisely when uv is the cheapest edge incident to either of them. There are $(n-1) + (n-1) - 1 = 2n - 3$ such edges because uv is incident to both cities. A uniformly random order on all edges induces a uniformly random relative order on this subset. Therefore

$$\mathbb{P}(u \text{ and } v \text{ choose each other}) = \frac{1}{2n-3}.$$

Expected number of components

For a fixed u, the events that u and v choose each other are disjoint as v ranges over the other $n-1$ cities. It follows that

$$\mathbb{E}[J_u] = \mathbb{P}(J_u = 1) = \frac{n-1}{2n-3}.$$

Linearity of expectation applies even when the indicators J_u are dependent. For example, with three cities, the endpoints of the cheapest edge always choose each other, so exactly two indicators are 1. Any two indicators therefore determine the third. Taking expectations in the counting identity yields

$$\mathbb{E}[C_n] = \frac{1}{2}\sum_u \mathbb{E}[J_u] = \frac{n(n-1)}{2(2n-3)} \quad \text{for } n \geq 2.$$

Answer for 100 cities $\mathbb{E}[C_{100}] = \dfrac{4950}{197} \approx 25.127.$

For $n = 2$ or $n = 3$, there is exactly one component, since each component contains at least two cities. As n tends to infinity, $\mathbb{E}[C_n]/n$ tends to $1/4$. If the rule is extended to $n = 1$ by letting the lone city make no choice, the network has one component.

Comparison with the official solution

The official solution shows that the built network is a forest. On any hypothetical undirected cycle, each endpoint of the most expensive edge is incident to a cheaper edge of the cycle. Neither endpoint would choose that edge. If M edges are chosen by both endpoints, the n choices produce $n-M$ distinct edges. Subtracting the number of edges from the number of vertices in a forest gives the number of components, which here is M.

Both proofs count the pairs that choose each other. The forest identity gives the total count, and following the arrows identifies the pair in each component. The official solution correctly computes the probability $1/197$ for a fixed pair when $n = 100$ and sums over all $\binom{100}{2}$ pairs.

Official solution, pages 10 and 11 · HMMT November 2026 archive