2026 Utah Math Olympiad

Alternative Solutions and Notes

April 11, 2026

Page references in the notes refer to the organizers' solution booklet.

Sources Contest problems · Official solutions

Corrections and clarifications

- Problem 2. The move used to establish the lower bound may be unavailable at the boundary.
- Problem 4. The bound $\log a_i \geq i$ fails for the sequence used in Solution 2.
- Problem 5. Lemma 6 uses the wrong index conditions in the two cases of the swap argument.
- Problem 6. In one configuration, the four lines in a group are distinct and parallel. In another, some of the indexed lines coincide.

Problem 1 A factorial equation

At $n = 0$, the two sides of $n! + 1 = (kn + 1)^2$ are 2 and 1, respectively. For $n \geq 1$, expanding the square, dividing by n^2, and using $1 \leq k \leq 10$ shows that any solution satisfies

$$\frac{(n-1)!}{n} = k^2 + \frac{2k}{n} \leq 100 + \frac{20}{n}.$$

For $n \geq 2$, successive values on the left have ratio $n^2/(n+1) > 1$, so the left side increases strictly. The right side decreases with n. At $n = 7$ we have $6!/7 = 100 + 20/7$, so the inequality fails for every $n > 7$.

For $n = 1, \ldots, 7$, the values of $n! + 1$ are $2, 3, 7, 25, 121, 721, 5041$, respectively. The only squares in this list are $25 = 5^2$, $121 = 11^2$, and $5041 = 71^2$. Solving $4k + 1 = 5$, $5k + 1 = 11$, and $7k + 1 = 71$ gives exactly the three pairs $(n, k) = (4, 1), (5, 2), (7, 10)$.

Notes on the official solutions

Solution 1 on p. 1 rules out $n \geq 8$ and then lists $0 \leq n \leq 8$ as the remaining range. The upper endpoint should be 7. In Solution 2 on p. 1, the notation $v_p(x)$ in Legendre's formula denotes the largest *exponent* e for which $p^e \mid x$. These two corrections leave the three published pairs unchanged.

Problem 2 Pursuit on a grid

Number the rows and columns from 1 to N so that James starts at $(1,1)$. The distinct starting positions require $N \geq 2$. Let $B_t = (x_t, y_t)$ be Brutus's position immediately after its tth move, with B_0 denoting its starting position.

James can catch Brutus in at most $N-1$ moves. If play continues after Brutus's tth move, James moves to

$$J_t = \big(\min\{t+1, x_t\}, \min\{t+1, y_t\}\big).$$

This formula also gives $J_0 = (1,1)$. From one turn to the next, each argument of each minimum changes by at most one, so each coordinate of J_t also changes by at most one. Moreover, both coordinates of J_{t-1} are at most t. If $J_t = J_{t-1}$, both minima defining J_t are strictly less than $t+1$, so $B_t = J_{t-1}$. In that case Brutus has already been caught on its own move. Whenever play continues, at least one coordinate of J_t changes. James can make this move, including when both coordinates change. At $t = N-1$, both coordinates of B_t are at most N, so $J_t = B_t$.

Brutus can force James to make at least $N-1$ moves. Choose a coordinate in which Brutus starts at some value $u \geq 2$. Brutus increases this coordinate by one on each turn until it reaches N. Thereafter it moves only along the other coordinate, where a legal move is always available because $N \geq 2$. After Brutus's tth move, the chosen coordinate equals $\min\{N, u+t\}$. For $1 \leq t \leq N-1$, this value exceeds t, the largest value James can reach in that coordinate before making his tth move. For $1 \leq t \leq N-2$, it also exceeds $t+1$, the largest value James can reach after his tth move. Thus Brutus cannot be caught before James makes his $(N-1)$st move.

The two bounds agree for every legal starting position of Brutus. Under optimal play, the expected number of moves James makes is therefore $N-1$.

Notes on the official solutions

In the upper-bound proof on p. 3, the induction needs to cover the case when the players' values in one coordinate agree and Brutus moves along that coordinate. For example, from $J = (1,1)$ and $B = (2,1)$, Brutus can move to $(2,2)$, breaking equality in the second coordinate. James can restore equality by moving one unit in that coordinate.

In the lower-bound proof on pp. 3–4, the published argument claims that Brutus can increase the separation in the chosen coordinate to 2. This step fails when Brutus is already at the boundary in that coordinate. For example, if $N = 2$, $J = (1,1)$, and $B = (2,1)$, the separation in either coordinate is at most 1. Brutus can still move to $(2,2)$ and force James to make one move. The strategy above, which keeps the chosen coordinate fixed once it reaches the boundary, proves the required lower bound for all starting positions.

Problem 3 Counting cube arrangements

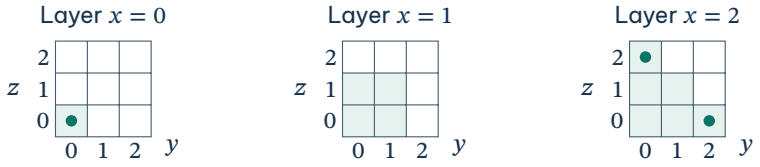
There are **52 arrangements**. We count them by considering the three layers from left to right.

Let x, y, z range from 0 to 2, increasing from left to right, from front to back, and from bottom to top, respectively. In the front view, the square (x, z) is orange if and only if $z \le x$. In the top view, the square (x, y) is orange if and only if $y \le x$. In the left view, the column corresponding to y has index $2 - y$ from the left, so the square (y, z) is orange if and only if $y + z \le 2$. Thus a position (x, y, z) can contain a cube only if

$$y \le x, \qquad z \le x, \qquad y + z \le 2.$$

Placing cubes only at these positions leaves every white square empty in each projection. The remaining requirement is that each line of sight corresponding to an orange square contain at least one cube.

In the layer $x = 0$, only the position $(0, 0, 0)$ can contain a cube. The orange square $(x, z) = (0, 0)$ in the front view forces a cube at that position. Similarly, the orange square $(x, z) = (2, 2)$ in the front view forces a cube at $(2, 0, 2)$, and the orange square $(x, y) = (2, 2)$ in the top view forces a cube at $(2, 2, 0)$.



Shading marks admissible positions. Dots mark forced cubes.

In the layer $x = 1$, the four positions with $(y, z) \in \{0, 1\}^2$ form a 2×2 grid. The front and top views require a cube in each row and each column. In the layer $x = 2$, there are four positions with $(y, z) \in \{0, 1\}^2$ in addition to the two forced positions. The forced cubes account for the orange squares with $y = 0, 2$ in the top view and with $z = 0, 2$ in the front view. Thus those two views require only that the layer $x = 2$ contain at least one cube with $z = 1$ and at least one cube with $y = 1$.

The three forced cubes account for the orange squares $(0, 0), (0, 2), (2, 0)$ in the left view. Each of the remaining orange squares $(0, 1), (1, 0), (1, 1)$ must come from a cube in the layer $x = 1$ or $x = 2$. Let E consist of those three positions that are empty in the layer $x = 2$. The layer $x = 1$ must contain a cube at every position in E. The requirements for $z = 1$ and $y = 1$ in the layer $x = 2$ leave five possibilities for E.

$$E = \varnothing, \quad \{(0, 1)\}, \quad \{(1, 0)\}, \quad \{(1, 1)\}, \quad \{(0, 1), (1, 0)\}.$$

If $E = \varnothing$, the grid in the layer $x = 1$ can have no empty squares, one empty square, or two empty squares at opposite corners. The corresponding numbers of choices are 1, 4, and 2, for a total of 7. If E consists of a single square, exactly two of these seven grids leave that square empty, so there are 5 choices. If $E = \{(0, 1), (1, 0)\}$, the two cubes at opposite corners satisfy the row and column requirements, and the other two corners can be filled independently. This gives 4 choices. Finally, each of the three lines of sight through $(2, 0, 0)$ already contains a forced cube, so that position contributes a factor of 2. The number of arrangements is therefore $2(7 + 3 \cdot 5 + 4) = 52$.

Notes on the official solutions

On page 5, Method 1 uses the condition $c_0 = 1$ in both branches of Case 1. The second branch should read $c_0 = 0$, which is when the conditions $\max(a_0, c_0) = \max(c_0, d_0) = 1$ force $a_0 = d_0 = 1$. The four groups listed cover all possibilities. The reference to five cases is a counting slip. On page 6, when four vertices of the hexagon are colored, the expression $\binom{6}{2} - 6$ counts choices of two *uncolored* vertices and excludes the six adjacent pairs. These slips leave the count of 52 intact.

Problem 4 The sum-of-two property

The assertion is **false**. We use base-4 expansions to construct a sequence with the sum-of-two property whose ratios a_i/i are unbounded.

Let U be the set of nonnegative integers whose base-4 digits are all 0 or 1. Let $a_0, a_1, \ldots$ list the elements of $U \cup 2U$ in increasing order, where $2U = \{2u : u \in U\}$. Every base-4 digit d can be written uniquely as $d = \varepsilon + 2\delta$ with $\varepsilon, \delta \in \{0, 1\}$. Given $n = \sum_{j=0}^{m} d_j 4^j$, applying this decomposition to each d_j yields

$$n = \sum_{j=0}^{m} \varepsilon_j 4^j + 2 \sum_{j=0}^{m} \delta_j 4^j.$$

The first summand belongs to U and the second to $2U$. Hence every nonnegative integer is a sum of two terms of the sequence.

For each $m \geq 1$, there are exactly 2^m elements of U below 4^m, since each of the m digit positions has two choices. The elements of $2U$ have digits 0 or 2, so exactly 2^m of them lie below 4^m. The two sets have only 0 in common. Since 4^m belongs to U, its index in the sequence is $2^{m+1} - 1$. It follows that

$$\frac{a_{2^{m+1}-1}}{2^{m+1}-1} = \frac{4^m}{2^{m+1}-1} > 2^{m-1}.$$

These ratios are unbounded, so the constant K in the assertion cannot exist.

Notes on the official solutions

On page 7, Solution 2 correctly derives the bound

$$a_i \geq \frac{(i+1)\sqrt{\log a_i}}{C}$$

for all sufficiently large i, where a_i has index i in the increasing sequence of sums of two squares. The next step incorrectly uses $\sqrt{i}$ as a lower bound for $\sqrt{\log a_i}$. The sequence contains the $i + 1$ distinct squares $0^2, 1^2, \ldots, i^2$, since $j^2 = j^2 + 0^2$. Consequently $a_i \leq i^2$, so $\log a_i \leq 2 \log i < i$ for all sufficiently large i.

The preceding bound suffices. Since the terms are distinct nonnegative integers in increasing order, $a_i \to \infty$ and hence $\sqrt{\log a_i}/C \to \infty$. It follows that $a_i/i \to \infty$, which establishes the proposed counterexample.

On page 6, the initial terms of both sequences contain index slips. In Solution 1, $a_3 = 4$ and $a_4 = 9$. In Solution 2, $a_5 = 8$. The note on page 7 states an equivalence with Goldbach's conjecture. The published solution leaves the implication from the sum-of-two property to Goldbach's conjecture unproved. For example, a representation $n = (p + 1) + (q + 1)$ with primes p, q gives a prime sum for $n - 2$. The two solutions do not use this equivalence.

Problem 5 Ordering the logs

Place $x_{n+1}, x_{n+2}, \dots, x_{2n}$ in the odd positions in increasing order and $x_n, x_{n-1}, \dots, x_1$ in the even positions in decreasing order. This order and the order obtained by swapping its first two logs are the two maximizers.

Take Fiona's starting point to be the center of the stump, as indicated in the contest diagram. Let d_j be her distance from the center after j rounds, with two logs used per round. Write u_j, v_j for the lengths of the two logs in round j, in their order of use. The Pythagorean theorem gives

$$d_0 = 0 \quad \text{and} \quad d_j = \sqrt{(d_{j-1} + u_j)^2 + v_j^2}.$$

Swapping u_1 and v_1 preserves d_1. For $n \geq 2$, we show that the last round must use the longest log first and the shortest log second.

The shortest log goes last. If $u_n < v_n$, swapping the two logs increases d_n^2 by the positive amount $2d_{n-1}(v_n - u_n)$. Suppose some log in an earlier round is shorter than v_n. Swapping it with v_n preserves the sum of squared lengths in

$$d_n^2 = \sum_{j=1}^{n} (u_j^2 + v_j^2) + 2\sum_{j=1}^{n} u_j d_{j-1}.$$

In the second sum, no d_{j-1} decreases, because the replacement in the earlier rounds is longer and the recurrence is increasing in each variable. No u_j decreases either. Moreover, d_{n-1} strictly increases, so the term $u_n d_{n-1}$ increases. Hence every maximizing order has $v_n = x_1$.

The longest log goes next to last. Shortening a log in the first $n-1$ rounds by $\delta > 0$ reduces d_{n-1} by less than δ. Indeed, in the round containing the shortened log, the vectors $(d_{j-1}+u_j, v_j)$ before and after the change differ in one coordinate and have both coordinates positive. The reverse triangle inequality bounds the difference between their lengths by δ, with strict inequality because the two vectors are not parallel. In each subsequent round, the same inequality shows that the difference cannot increase. If a log in an earlier round is longer than u_n by δ, swapping it with u_n reduces d_{n-1} by less than δ and increases u_n by exactly δ. Thus $d_{n-1} + u_n$ increases and v_n stays fixed, so d_n increases. It follows that $u_n = x_{2n}$.

Once the final pair is fixed, d_n is strictly increasing in d_{n-1}. Thus the preceding rounds must maximize the distance attainable with the remaining logs. Repeating the selection of the longest and shortest remaining logs leaves x_n, x_{n+1} for the first round. Both orders of this pair give the same d_1. A maximum exists because there are finitely many orders, and precisely the two orders stated above attain it.

Notes on the official solutions

In Lemma 6 of Solution 1 (p. 10), the two case conditions are interchanged. The case $i \neq 1$ allows $j = 1$, in which case the asserted inequality after the swap is $a_1' > a_1'$. Likewise, the case $j \neq 1$ allows $i = 1$ and leads to $a_2' > a_2'$. The comparison with a_{2j-1}' requires $j \neq 1$, and the comparison with a_{2i}' requires $i \neq 1$. The intermediate inequalities are $a_2 \geq a_{2i}$ and $a_{2j-1} \geq a_1$, respectively.

In Lemma 5 (p. 10), the functions evaluated on $2k + 2$ logs should carry the subscript $k + 1$. The swap there shows that the last log must be the shortest among those in even positions. The induction uses the fact that the prefix of a maximizing order is itself maximizing for its own logs, since the final distance is strictly increasing in the distance before the last round. In Lemma 3 (pp. 8–9), f_{2k-2} should be f_{k-1}. In the base case of Lemma 4 (p. 9), f_2^2 should be f_1^2.

Problem 6 Four tangent circles

We use the hypothesis that each $P_j \neq O_j$ exists and is unique. We treat the lines as a family indexed by (i, j), and a group with a common direction may contain coincident lines. For each permutation σ in the list id, $(12)(34)$, $(13)(24)$, $(14)(23)$, take the group of four lines $O_iP_{\sigma(i)}$ with $1 \leq i \leq 4$.

First, the tangency points are pairwise distinct. If three circles share a tangency point, every other point on their common tangent lies outside all three circles and has the same power with respect to each. This contradicts the uniqueness of the corresponding P_j. If two disjoint pairs share a tangency point, all four circles pass through that point, so at least three are tangent there.

Suppose one of the disks contains another. A disk of largest radius is either the larger disk in that pair or contains it. Otherwise it would be externally tangent to that larger disk and could touch the smaller one only at the same tangency point. For the same reason, the largest disk contains every remaining disk. Two disks inside the largest disk cannot be nested, since they would then touch the largest disk at the same point. Use signed radii r_i, with negative sign for the circle enclosing the others if there is one. Then $|O_i - O_j|^2 = (r_i + r_j)^2$ for $i \neq j$.

Write $O_i = (x_i, y_i)$ and set

$$v_i = \frac{(1, x_i, y_i, x_i^2 + y_i^2 - r_i^2)}{r_i}.$$

On $\mathbb{R}^4$, use the bilinear form

$$\langle (a, b, c, d), (a', b', c', d') \rangle = bb' + cc' - \frac{ad' + da'}{2}.$$

Expanding gives $\langle v_i, v_i \rangle = 1$ and

$$\langle v_i, v_j \rangle = \frac{r_i^2 + r_j^2 - |O_i - O_j|^2}{2r_ir_j} = -1 \quad \text{for } i \neq j.$$

If $\sum_j \lambda_j v_j = 0$, pairing with v_i gives $2\lambda_i - \sum_j \lambda_j = 0$ for every i. Summing the four equations and substituting back gives $\lambda_i = 0$ for every i. Hence $v_1, \ldots, v_4$ form a basis.

Set $H = (v_1 + v_2 + v_3 + v_4)/2$ and $w_j = v_j - H$. Since $\langle v_i, H \rangle = -1$, the matrix of pairings $\langle v_i, w_k \rangle$ is $2I$. Let $P_j = (X_j, Y_j)$ have common power ρ_j^2 with respect to the circles indexed by $i \neq j$. The power formula implies that $(1, X_j, Y_j, X_j^2 + Y_j^2 - \rho_j^2)$ is orthogonal to every v_i with $i \neq j$. The pairing matrix shows that the vectors with this property are exactly the scalar multiples of w_j.

Let π be the projection onto the first three coordinates. Then $\pi(v_i)$ is proportional to $(1, x_i, y_i)$ and $\pi(w_j)$ to $(1, X_j, Y_j)$. The points O_i and P_j are distinct by hypothesis for $i = j$, and because P_j lies outside ω_i for $i \neq j$. If a nonzero vector (a, b, c) belongs to $\text{span}(\pi(v_i), \pi(w_j))$, it is a linear combination of $(1, x_i, y_i)$ and $(1, X_j, Y_j)$. If $a \neq 0$, the point $(b/a, c/a)$ lies on O_iP_j. If $a = 0$, the vector (b, c) gives a direction of that line.

For the identity permutation, $H = v_i - w_i$ belongs to every $\text{span}(v_i, w_i)$. Since $\langle H, H \rangle = -2$, its first coordinate is nonzero. Indeed, if $a = 0$ then the form evaluates to $b^2 + c^2$. Thus $\pi(H)$ determines a common point. For a permutation $(ij)(kl)$, set $U = v_i + v_j - H$. Then

$$U = v_i + w_j = v_j + w_i = -(v_k + w_l) = -(v_l + w_k).$$

Since $\langle U, U \rangle = 2$, we have $\pi(U) \neq 0$. The vector $\pi(U)$ determines either a common point or a common direction of the four lines in this group.

The lines in each group cover all eight points because σ is a permutation. For each i, the four permutations send i to four different values, so every pair (i, j) is used exactly once.

Problem 6 Notes on the official solutions

The hypothesis on the points P_j. Pairwise tangency alone does not ensure that these points exist. For example, let ω_2 have center $(0,0)$ and radius 2, let ω_3, ω_4 have centers $(-1,0), (1,0)$ and radius 1, and let ω_1 have center $(0,4/3)$ and radius $2/3$. The four circles are pairwise tangent at six distinct points. The radical axes of the three pairs drawn from $\omega_2, \omega_3, \omega_4$ are $x=-2$, $x=2$, and $x=0$. Thus P_1 does not exist. In the next two examples, every triple of centers is noncollinear. Equating the three powers for each P_j therefore gives two independent linear equations.

Parallel lines in Solution 2 (pp. 22–23). On p. 22, Solution 2 invokes a converse of Pitot's theorem to infer the existence of a circle δ tangent to the four lines containing the sides of $O_1O_2O_4O_3$. This inference fails in the following example. Take

$$(O_1, O_2, O_3, O_4) = ((-2,-3), (-2,0), (-6,-3), (-6,0)),$$
$$(|r_1|, |r_2|, |r_3|, |r_4|) = (1,2,3,6).$$

The three inner circles are pairwise externally tangent, and each is internally tangent to ω_4. The quadrilateral $O_1O_2O_4O_3$ is a 4×3 rectangle and satisfies $O_4O_2 + O_2O_1 = O_4O_3 + O_3O_1 = 7$. Tangency to its two pairs of parallel sides would require δ to have both radius 2 and radius $3/2$. Solving the equations for equal powers gives

$$P_1 = (0,-6), \quad P_2 = (-3,-6),$$
$$P_3 = (0,-2), \quad P_4 = (-3,-2).$$

Their common powers are $36, 9, 4, 1$, respectively, all positive. The lines $O_1P_4, O_2P_3, O_3P_2, O_4P_1$ are distinct and parallel, with equations $x+y=c$ for $c=-5,-2,-9,-6$, respectively. These are the four lines that Solution 2 claims are concurrent on p. 23.

Coincident lines and the use of Desargues' theorem (p. 24). Consider the four externally tangent circles with

$$(O_1, O_2, O_3, O_4) = ((0,3/4), (0,5/12), (-1,0), (1,0)),$$
$$(r_1, r_2, r_3, r_4) = (1/4, 1/12, 1, 1).$$

Equating powers gives $P_1 = (0,1/5)$, $P_2 = (0,1/3)$, $P_3 = (1/8,1/2)$, and $P_4 = (-1/8,1/2)$. The common powers are $1/25, 1/9, 1/64, 1/64$, respectively. Both O_1O_2 and P_1P_2 are the line $x=0$, so the intersection point T in the Desargues argument is not uniquely defined. The lemma's noncollinearity hypotheses still hold. Indeed, for distinct i, j, k, the point P_k lies on the common tangent of ω_i, ω_j and is different from the tangency point, so it lies off O_iO_j. Also, P_iP_j is the common tangent of the other two circles, so it misses O_k. The last paragraph on p. 24 also misidentifies the triple defining P_3. The correct triple is $\omega_1, \omega_2, \omega_4$.

In this configuration, only three distinct lines in the family pass through O_1, namely $x=0$, O_1P_3, and O_1P_4. The other lines meet the vertical axis at heights in $\{1/5, 1/3, 5/12, 4/9, 4/7\}$. If each distinct line may be used only once, four disjoint groups with a line through O_1 are impossible. The statement should specify that the sixteen *index pairs* (i,j) are partitioned, with different pairs allowed to determine the same line.

A notation error in Solution 1 (p. 14). In the argument that the rectangle is a square, the text refers to the radius of ω_1', which is a line. The radius needed is that of the circle ω_3', which is tangent to both parallel lines ω_1', ω_2'.